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The Hannay-Berry Phase of the Vibrating Ring Gyroscop

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THE HANNAY-BERRY PHASE OF THE VIBRATING RING GYROSCOPE [*]

SEAN B. ANDERSSON [†] AND P.S. KRISHNAPRASAD [‡]

Abstract.
In an analysis published in 1890 G.H. Bryan investigated the retrograde precession of the nodal points in a vibrating, rotating shell and wrote down a formula relating the rate of precession to the rate of rotation. This effect has been utilized in the design of various vibratory gyroscopes including modern MEMS-based devices. Existing analyses model these systems with a pair of harmonic oscillators coupled through the Coriolis force (the normal mode method). In this work we utilize the theory of moving systems developed by Marsden, Montgomery, and Ratiu to show that the nodal precession can be understood as a geometric phase with respect to the Cartan-Hannay-Berry connection. This approach allows us to explicitly characterize the simplifications of the linearizing assumptions common to previous analyses. Our results match those of Bryan for small amplitude vibrations of the ring. We use the inherently nonlinear nature of the moving systems approach to calculate a (small) correction to the rate of precession of the nodes.

Key words. geometric phase, holonomy, fiber bundle, vibratory gyroscope, MEMS, rotation sensing

AMS subject classifications. 53C29, 70G45, 70H05, 70K65,74F99, 74K10, 81Q70, 93C95

1. Introduction . In 1890 G.H. Bryan published a paper on the nature of the beats generated when a vibrating shell is rotated about its central axis [8]. The phenomenon he describes is quite easy to observe; simply take a wine glass, strike it to produce a clear tone, and then rotate it about its stem to produce audible beats. Bryan noticed that these beats are the result of a precession of the nodal points with respect to the shell itself and provided the following reasoning. Consider a ring or cylinder rotating about its central axis and vibrating with nodes at B,D,F, and H as indicated in the left-side image of Figure 1.1. The material points at A and E are moving towards the center O. This increases their actual angular velocity above that of the imposed rotation and gives them a relative angular acceleration in the direction of rotation as represented by the arrows at A and E in the right side image of Figure 1.1. Similarly, the material points at C and G are moving outwards and thus their angular velocity is reduced. Those at B and F are moving with greater total angular velocity than the rest and thus experience a relative outwards acceleration due to a greater centrifugal force. Finally, the material points at D and H are moving with the least angular velocity and thus experience a relative acceleration inwards. Comparing the arrows in the two images of Figure 1.1 reveals that the effect of these relative accelerations is to cause retrogade motion of the nodes relative to the ring. Using classical variational techniques Bryan derived a linearized partial differential equation describing the behavior of the system, found a formula for the rate of precession and

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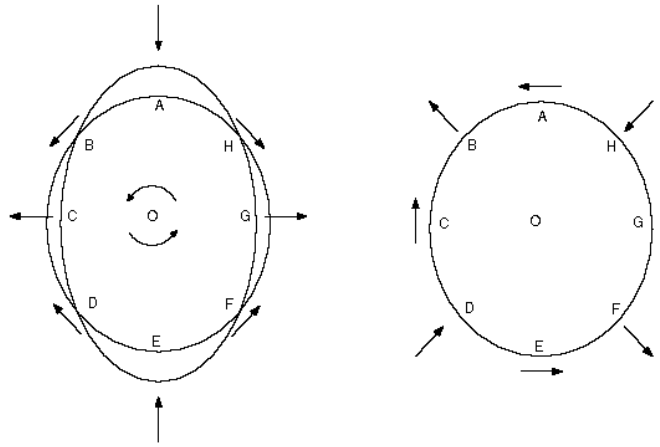
FIG. 1.1. *Nodal precession in cylinder or ring (Figure from [8])*

discovered that this rate is proportional to the rate of platform rotation.

Due to the immense number of potential applications, research on gyroscopes has been active for many years. Devices have been proposed, analyzed, and produced using a variety of materials and techniques. Because they lack rapidly spinning parts, have low power requirements, and are inherently scalable, vibratory gyroscopes have become particularly popular [21]. One of the most successful initial designs was Delco's Hemispherical Resonator Gyroscope (HRG) [15] due to Loper and Lynch, which was able to achieve performance levels equal to the best ring laser gyroscopes. This design, shown diagrammatically in Figure 1.2, consists of a quartz hemispherical resonator supported on a central stem and contained inside an evacuated housing. As predicted by Bryan, the nodal points of the vibration in the hemisphere precess
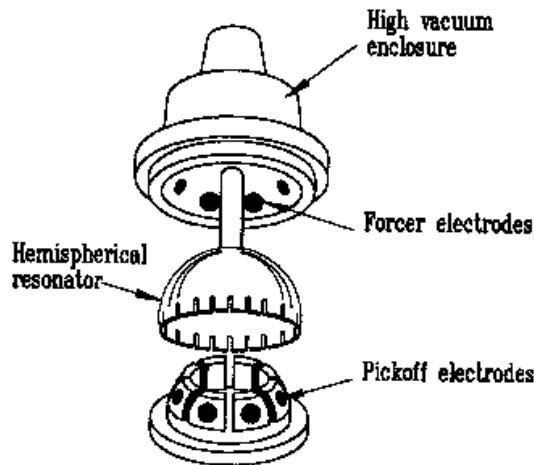

FIG. 1.2. *Delco Hemispherical Resonator Gyroscope (Image from [14])*

with respect to the shell as the device is rotated. The HRG is driven into elliptical

vibration as shown in Figure 1.3 and the resulting precession rate is about 0.3 of that of the imposed rotation. The device operates over a wide temperature range, has high operating acceleration ranges, low accleration sensitivies, and negligible magnetic sensitivity [16]. Similar ideas were used in the design of a vibrating disc gyroscope by
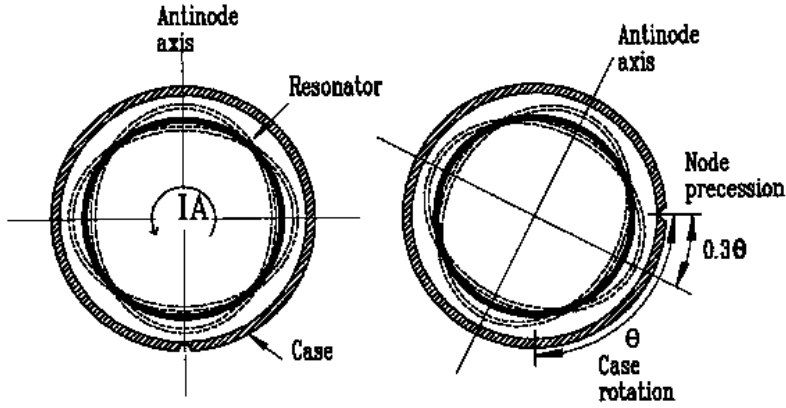
FIG. 1.3. *Precession of Delco HRG (Image from [14] following [15])*

Burdess and Wren [10].

With the explosion of MEMS technology constant innovations are resulting in smaller, cheaper, and more accurate devices. Existing MEMS-based devices include tuning-fork [5] and vibrating-ring designs [4, 22] such as the one shown in Figure 1.4 (provided by Douglas Sparks of Delco Automotive Systems). Additional designs proposed include a vibrating cylinder [9, 28] and a surface acoustic wave generator [27].

FIG. 1.4. *MEMS Gyroscope (Image courtesy of Douglas Sparks of Delco Automotive Systems)*

These gyroscopes all take advantage of the same physical effect, the Coriolis force

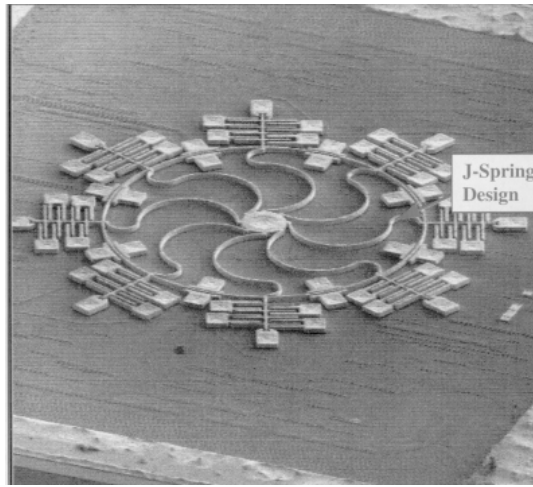arising from the non-inertial character of the rotating frame of the system [14]. Modern analyses are linear in nature and view the Coriolis force as providing a coupling between two vibratory modes of the structure [25]. It is desirable, however, to have a method which, at least in principle, can be extended to a nonlinear theory and which provides a unified setting for understanding a variety of systems in which the Coriolis force plays a role. Motivated by these considerations we are led to an approach developed by Marsden, Montgomery, and Ratiu based on modern developments in geometric mechanics. This method is known as the moving systems approach [18]. The technique, descending from the classical work of E. Cartan [11], describes the effect of imposed motion on a system as a geometric phase with respect to a particular Ehresmann connection called the Cartan-Hannay-Berry connection. This geometric phase is termed the Hannay-Berry phase. In previous work we have used this approach to determine that the Hannay-Berry phase in a rotating, equal-sided, spring-jointed, four-bar mechanism is zero [3]. (See also [2].)

Interest in the effect of geometric phases in physical systems was spurred by work of Michael Berry in the early 1980's [6, 7]. Berry's initial result, surprising in its simplicity, has sparked a great deal of research that continues to this day (see, e.g. the compendium volume of Wilczek and Shapere [24]).

While existing techniques have proved to be effective, as evidenced by the various devices constructed from those principles, it is to be expected that a deeper understanding will emerge by appealing to a nonlinear, geometric approach directed at more accurate constitutive models. The fact that Bryan's result is geometrical in nature is intriguing and suggests that these modern tools may prove useful. In this paper we begin this study by applying the moving systems approach to the vibrating ring and show that Bryan's result can be interpreted as the geometric phase. The main contribution of this work is to provide, through the use of geometric techniques, a clear understanding of the role of the linearization assumptions common to previous analyses and to make explicit the simplifications that result. Furthermore we calculate the effect of the neglected terms by deriving a formula for a correction term to the rate of precession of the nodes of vibration. Finally, this work illustrates, through the reasonably complex example of the ring gyroscope, how the methods of modern geometric mechanics are useful in understanding physical effects in systems which are useful for sensing and as a result hopefully motivates new applications of the ideas.

The remainder of the paper is organized as follows. In the next section we provide a brief background the proper mathematical setting for geometric phases, namely that of connections on fiber bundles. In Section 3 we summarize the moving systems approach of Marsden, Montgomery, and Ratiu. In Section 4 we apply the technique to the vibrating ring and derive the Hannay-Berry phase which is then compared in Section 5 to the results of Bryan. The imposed rotation on the ring gives rise to nonlinear terms in the equations of motion even when a linear approximation of the nominal (non-moving) ring is used and in Section 6 we calculate a correction term incorporating these effects. We conclude in Section 7 with a few comments on the results and on future work.

2. Background . The natural mathematical framework from which to approach geometric phases is that of connections on fiber bundles. In this section we present a review of the necessary background on fiber bundles, connections, and holonomy. Additional references for this material include [12, 19] and references therein.

2.1. Fiber bundles and Ehresmann connections. A **fiber bundle** is defined as follows. Let P, F, and B be manifolds referred to as the **total space** (or

bundle space), the **fiber**, and the **base space** respectively. Let $\pi : P \to B$ be a surjective submersion. We require that P be locally a product space, that is, for every $b \in B$ there is a neighborhood U of b such that $\pi^{-1}(U)$ is diffeomorphic to $F \times U$. The fiber over b, $\pi^{-1}(b)$, is a diffeomorphic copy of the fiber F for every $b \in B$. See Figure 2.1. The bundle is denoted by the triple (P, F, B) or by the projection map
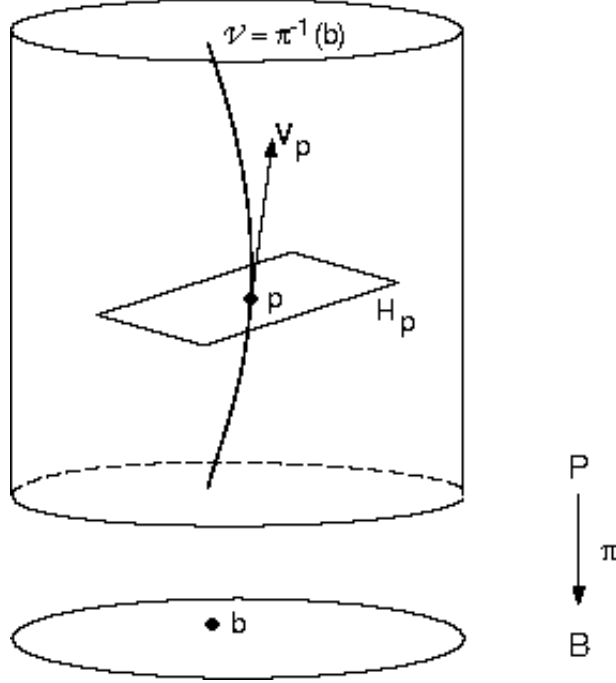

FIG. 2.1. *Fiber bundle, fiber over b, and splitting of the tangent space*

$\pi : P \to B$. If the bundle is globally a product bundle, $P = F \times B$, then it is called a trivial fiber bundle.

Given $p \in P$, there is a natural subspace of T_pP (the tangent space to P at p) called the vertical space at p, denoted by V_p and defined by $V_p \triangleq \ker T_p\pi$. Here $T_p\pi$ is the linearization of the projection map evaluated at p. The union of these subspaces over all p is called the vertical subbundle V, i.e. $V \triangleq \cup_{p \in P}V_p$.

DEFINITION 2.1. *An **Ehresmann connection** A on P is a vertical-valued one-form on P satisfying:*

1. $A_p : T_pP \to V_p$ is a linear map.

2. A_p is a vertical projection. That is, $A_p(v) = v \quad \forall v \in V_p$.

■

The connection defines a horizontal space $H_p \triangleq \ker A_p$ at each point $p \in P$. The conditions in the definition imply $T_pP = V_p \oplus H_p$ and thus the connection gives a splitting of the tangent space at each point p into a vertical and a horizontal part (see again Figure 2.1). The union of these subspaces over all p is called the horizontal subbundle, i.e. $H = \cup_{p \in P}H_p$.

Given an Ehresmann connection A, a point $p \in P$, and a tangent vector $w \in T_{\pi(p)}B$, define the horizontal lift of w to T_pP as the unique tangent vector in H_p that

projects to w under $T_p\pi$. We call this lift hor_p. The lift of w can be found by

$$\mathrm{hor}_p w = \widetilde{w} - A_p(\widetilde{w}) \tag{2.1}$$

where $\widetilde{w} \in T_p P$ is an arbitrary vector satisfying $T_p\pi(\widetilde{w}) = w$.

LEMMA 2.2. *The map* hor_p *is well-defined.*

Proof. Let $\widetilde{u}_1, \widetilde{u}_2 \in T_p P$ be tangent vectors such that

$$T_p\pi(\widetilde{u}_1) = T_p\pi(\widetilde{u}_2) = u.$$

Notice that

$$T_p\pi\left(\widetilde{u}_1 - \widetilde{u}_2\right) = T_p\pi\left(\widetilde{u}_1\right) - T_p\pi\left(\widetilde{u}_2\right) = u - u = 0$$

and thus $\widetilde{u}_1 - \widetilde{u}_2$ is a vertical vector. Let

$$\mathrm{hor}_p^1[u] = \widetilde{u}_1 - A_p\widetilde{u}_1 \qquad \text{and} \qquad \mathrm{hor}_p^2[u] = \widetilde{u}_2 - A_p\widetilde{u}_2.$$

Then

$$
\begin{aligned}
\mathrm{hor}_p^1[u] - \mathrm{hor}_p^2[u] &= (\widetilde{u}_1 - A_p\widetilde{u}_1) - (\widetilde{u}_2 - A_p\widetilde{u}_2) \\
&= (\widetilde{u}_1 - \widetilde{u}_2) - A_p\left(\widetilde{u}_1 - \widetilde{u}_2\right) \\
&= (\widetilde{u}_1 - \widetilde{u}_2) - (\widetilde{u}_1 - \widetilde{u}_2) = 0.
\end{aligned}
$$

□

It should be noted that while here we have defined the Ehresmann connection as a vertical-valued one-form and derived the horizontal space and horizontal lift, one can also begin with the definition of the horizontal space or the horizontal lift and define the other two objects. See [18] for details.

2.2. Parallel transport and holonomy. Given an Ehresmann connection A on a fiber bundle, define parallel transport with respect to A along a curve lifted from the base space in the following way. Let $b(t)$, $t \in [0,1]$, be a piecewise differentiable curve in B. The horizontal lift of $b(t)$ with respect to A is the curve $p(t)$ in P such that $\pi(p(t)) = b(t)$ and that the tangent vector $\frac{dp(t)}{dt}$ is horizontal for each $t \in [0,1]$. We have the following proposition from [18].

PROPOSITION 2.3. [18] *Given a curve* $b(t)$, $t \in [0,1]$, *in* B *and a point* $p_0 \in \pi^{-1}(b(0))$, *there exists a unique locally defined horizontal lift* $p(t)$ *of* $b(t)$ *to* P *satisfying* $p(0) = p_0$ *if* P *is a locally trivial fiber bundle.*

Proof. Since P is locally a trivial bundle we can write locally $p = (b, f)$ and $\dot{p} = (u, v)$ for $b \in B$, $f \in F$, $u \in T_b B$, and $v \in T_f F$. We can then write the connection one form as

$$
\begin{aligned}
A(b,f)(u,v) &= (0, v) + A(b,f)(u, 0) \\
&\triangleq (0, v + \lambda(b,f)u).
\end{aligned}
$$

(u,v) is horizontal if and only if $A(b,f)(u,v) = 0$ and thus for a horizontal tangent vector $v = -\lambda(b,f)u$. If $b(t)$ is a path in B denote $p(t) = (b(t), f(t))$ where $f(t)$ is the solution to the ordinary differential equation

$$\frac{df}{dt}(t) = -\lambda(b(t), f(t))\dot{b}(t), \quad f(0) = f_0 \text{ where } p_0 = (b_0, f_0). \tag{2.2}$$

Then by local existence and uniqueness for ordinary differential equations this defines $f(t)$ and thus $p(t)$ for small t. If $p(t)$ can be extended for all $t \in [0,1]$ the connection is called *complete*. $\square$

Given any curve $b(t)$ in B, $t \in [0,1]$, and an Ehresmann connection A, the **parallel transport operator** τ_b is defined as

$$(2.3) \qquad \tau_b : \pi^{-1}(b(0)) \to \pi^{-1}(b(1)), \qquad \tau_b(p(0)) = p(1)$$

where $p(0) \in \pi^{-1}b(0)$ and $p(t)$ is the horizontal lift of $b(t)$ with respect to A starting at $p(0)$.

By the uniqueness of the horizontal lift, τ_b is a bijection from $\pi^{-1}(b(0))$ to $\pi^{-1}(b(1))$ and by the smooth dependence of solutions of ordinary differential equations on initial conditions it is a diffeomorphism.

Let b_0 be an arbitrary point of B and let C_{b_0} be the set of all closed curves at b_0, that is all $b(t)$ such that $b(0) = b(1) = b_0$. The diffeomorphism of $\pi^{-1}(b_0)$ onto itself given by parallel transport along $b(t)$ is called the **holonomy** of the path $b(t)$. Let Φ_{b_0} be the collection of all parallel transport operators over C_{b_0} and define the group operation as composition. Φ_{b_0} then forms a group, called the holonomy group at b_0. (Assuming B is connected, it is easy to see that Φ_{b_0} and Φ_{b_1} are conjugate for any two $b_0, b_1 \in B$. Thus if B is connected we have simply Φ, the holonomy group of the connection.)

DEFINITION 2.4. *Given a bundle $\pi : P \to B$, a connection on the bundle, and a closed curve $b(t)$ in the base space, the* **geometric phase** *is the holonomy along the curve $b(t)$.* $\blacksquare$

The geometric phase can be calculated by solving the ordinary differential equation given in equation (2.2).

3. The moving systems approach . Inspired by classical examples such as the Foucault pendulum and the ball in a hoop, one is led to consider the question of variations in phase space as a parameter in a mechanical system is slowly varied along a closed path in parameter space, as in a moving system. Examples in quantum physics, optics, and other settings [6, 26, 24] reveal that the essential calculation is a geometric one and is in fact captured by a holonomy.

Marsden, Montgomery, and Ratiu have developed a modern geometric approach to understanding moving systems that utilizes the tools of Ehresmann connections on fiber bundles [18]. Here we present a brief synopsis of their approach. Let S be a Riemannian manifold and let M be the space of embeddings of a manifold Q into S. We think of S as the ambient space in which Q is being moved and of Q as the configuration space for a system of interest. A tangent vector to M at m is a map $u_m : Q \to TS$ such that $u_m(q) \in T_{m(q)}S$. Given a tangent vector $u_m(q)$ one can construct a tangent vector to T_qQ as follows. Relative to the metric on S, orthogonally project $u_m(q)$ to $T_{m(q)}m(Q) \subset (T_qm)(T_qQ)$, denote this vector $u_m^T(q)$, and then pull-back $u_m^T(q)$ by Tm^{-1} to T_qQ. This natural construction defines an Ehresmann connection on the product bundle $\pi : Q \times M \to M$ as follows.

DEFINITION 3.1. [18] *The* **Cartan connection** *on $\pi : Q \times M \to M$ is given by the vertical-valued one-form γ_c defined by*

$$(3.1) \qquad \gamma_c(q, m)(v_q, u_m) = (v_q + (T^{-1}m \circ u_m^T)q, 0).$$

$\blacksquare$

The Cartan connection induces a connection on $\rho : T^*Q \times M \to M$ as follows.

DEFINITION 3.2. [18] *The* **induced Cartan connection** *on* $\rho : T^*Q \times M \to M$ *is given by the vertical-valued one-form* γ_o *defined by*

$$(3.2) \qquad \gamma_o(\alpha_q, m)(U_{\alpha_q}, u_m) = \left(U_{\alpha_q} + X_{\mathcal{P}(u_m)}(\alpha_q), 0 \right)$$

where $\mathcal{P}(u_m)$ *is the function defined by*

$$(3.3) \qquad (\mathcal{P}(u_m))\,\alpha_q = \alpha_q \cdot (T^{-1}m \circ u_m^T)(q)$$

and $X_{\mathcal{P}(u_m)}$ *is the Hamiltonian vector field of* $\mathcal{P}(u_m)$. ■

To separate the effects of the imposed motion on the system (as defined by the embeddings m_t) from the nominal dynamics (when the imposed motion is zero) we use the ideas of averaging. Abstractly, we assume we are given a left action of a Lie group G on T^*Q. The average of the connection form γ is defined by

$$(3.4) \qquad \langle \gamma \rangle = \frac{1}{|G|} \int_G g^*(\gamma)\,dg$$

where dg is a left Haar measure and $|G|$ is the total volume of G. From this we have the following definition.

DEFINITION 3.3. [18] *The* **Cartan-Hannay-Berry connection** *on* $\rho : T^*Q \times M \to M$ *is given by the vertical-valued one-form* γ *defined by*

$$(3.5) \qquad \gamma(\alpha_q, m)(U_{\alpha_q}, u_m) = \left(U_{\alpha_q} + X_{\langle \mathcal{P}(u_m) \rangle}(\alpha_q), 0 \right)$$

where $\langle \cdot \rangle$ *denotes the average with respect to the action of the Lie group* G. ■ In [18] Marsden, Montgomery, and Ratiu show that this is an Ehresmann connection. The horizontal lift of a vector field Z on M relative to γ is

$$(3.6) \qquad (hor\,Z)(\alpha_q, m) = \left(-X_{\langle \mathcal{P}(Z(m)) \rangle}(\alpha_q), Z(m) \right).$$

DEFINITION 3.4. [18] *The holonomy of the Cartan-Hannay-Berry connection is called the* **Hannay-Berry phase** *for a moving system.* ■

4. The rotating, vibrating ring . In this section we derive the Hannay-Berry phase for the vibrating ring. Using the moving systems approach we find an explicit formula for the phase shift under linearizing assumptions and show that this result matches that of G.H. Bryan [8]. In this work we are interested in the effects of the imposed rotatory motion and as a consequence choose to simplify the analysis of the ring dynamics by assuming the ring has no cross-sectional area. This choice also allows a direct comparison to the results derived by Bryan. A more comprehensive treatment based on the geometrically exact theory of rods could be developed to understand the detailed dynamics of the ring itself (see, for example, [20]).

Consider a thin ring of length L and line density σ. The body is given by $\mathcal{B} = \{b : b \in [0, L]\}$. Let θ be the mapping given by

$$\theta : \mathcal{B} \to S^1$$
$$b \mapsto \left(\frac{2\pi}{L} \right) b$$

allowing us to parametrize the ring by $\theta \in [0, 2\pi]$. We define the reference configuration to be a circular ring of radius a centered on an inertial reference frame and $(w(\theta), \gamma(\theta))$ to be the radial and angular deformations from this reference respectively.

To maintain integrity of the ring we require $w(0) = w(2\pi)$ and $\gamma(0) = \gamma(2\pi)$. In standard cylindrical coordinates the configuration of the ring is given by $(a+w(\theta), \theta+\gamma(\theta))$. Let $\mathcal{C}$ be the space of all smooth deformations of the ring. (We do not discuss here the explicit infinite dimensional manifold structure for $\mathcal{C}$ and associated structures, although it is standard as in [17]). Since we are interested in imposed rotational movements of the ring (as a sensor), we split $\gamma(\theta) = \psi + \alpha(\theta)$ with $\alpha(0) = \alpha(2\pi)$ where ψ, independent of θ, is a global rotation.

We now use the following argument of Rayleigh [23]. Since the ring is thin the forces resisting bending are small in comparison to those which resist extension. In the limiting case of an infinitely thin ring the flexural vibrations become independent of any extension of the circumference as a whole and one may assume that each part of the circumference retains its natural length throughout the motion. Under this condition we say the ring is *inextensible*. Viewing the deformed ring as a curve in $\mathbb{R}^2$, a point on the curve is given in Cartesian coordinates by

$$(4.1) \qquad \begin{pmatrix} x(\theta) \\ y(\theta) \end{pmatrix} = \begin{pmatrix} (a + w(\theta)) \cos(\theta + \gamma(\theta)) \\ (a + w(\theta)) \sin(\theta + \gamma(\theta)) \end{pmatrix}.$$

Equating the lengths of an arbitrary section of the circumference of the reference configuration to the length of the same section in the deformed configuration yields

$$\int_{\theta_1}^{\theta_2} a \, d\theta = \int_{\theta_1}^{\theta_2} \sqrt{\left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2} \, d\theta$$

$$(4.2) \qquad = \int_{\theta_1}^{\theta_2} \sqrt{(a + w(\theta))^2 \left(1 + \frac{\partial \gamma}{\partial \theta}\right)^2 + \left(\frac{\partial w}{\partial \theta}\right)^2} \, d\theta.$$

Since θ_1 and θ_2 are arbitrary we have

$$0 = (a + w)^2 \left(1 + \frac{\partial \gamma}{\partial \theta}\right)^2 + \left(\frac{\partial w}{\partial \theta}\right)^2 - a^2$$

$$(4.3) \quad = 2aw + w^2 + 2(a^2 + 2aw + w^2)\frac{\partial \gamma}{\partial \theta} + (a^2 + 2aw + w^2)\left(\frac{\partial \gamma}{\partial \theta}\right)^2 + \left(\frac{\partial w}{\partial \theta}\right)^2.$$

From here on we assume the deformations are small and so we keep only terms to first order in equation (4.3). The inextensibility condition then requires that

$$(4.4) \qquad w = -a\frac{\partial \gamma}{\partial \theta} = -a\frac{\partial \alpha}{\partial \theta}.$$

From the above the space $\mathcal{C}$ is given by

$$(4.5) \qquad \mathcal{C} = \{(\psi, \alpha) | \psi \in S^1, \alpha : S^1 \to S^1, \alpha(0) = \alpha(2\pi), \alpha \text{ smooth}\}.$$

Any $W \in T_{(\psi, \alpha)}\mathcal{C}$ has the form (with overdot denoting the partial derivative with respect to time)

$$(4.6) \qquad W = \begin{pmatrix} \dot{\psi} \\ \dot{\alpha}(\theta) \end{pmatrix}.$$

The kinetic energy is easily verified to be

$$(4.7) \quad KE = \frac{1}{2} \int_0^{2\pi} \left[\left(1 - \frac{\partial \alpha}{\partial \theta}\right)^2 \left(\dot{\psi}^2 + 2\dot{\psi}\dot{\alpha}(\theta) + \dot{\alpha}^2(\theta)\right) + \left(\frac{\partial \dot{\alpha}}{\partial \theta}\right)^2\right] \sigma a^3 \, d\theta$$

where equation (4.4) has been used to express w in terms of α. This defines an inner product on $\mathcal{C}$ given by

$$(W_1, W_2) = \int_0^{2\pi} \left[\left(1 - \frac{\partial \alpha}{\partial \theta}\right)^2 \left(\dot\psi_1 \dot\psi_2 + \dot\psi_2 \dot\alpha_1(\theta) + \dot\psi_1 \dot\alpha_2(\theta) + \dot\alpha_1(\theta)\dot\alpha_2(\theta)\right) \right.$$

(4.8)
$$\left. + \frac{\partial \dot\alpha_1}{\partial \theta} \frac{\partial \dot\alpha_2}{\partial \theta} \right] \sigma a^3 d\theta.$$

As in Bryan, we take the potential energy due to the bending of the ring to be proportional to the change in curvature squared of the ring. That is

(4.9)
$$V = \frac{\beta}{2} \int_0^{2\pi} \left(\kappa_\alpha(\theta) - \kappa_{\alpha \equiv 0}(\theta)\right)^2 a d\theta$$

where $\kappa_\alpha(\theta)$ is the curvature of the ring at the material point θ under the deformation α and β is a material constant. For a curve $(r(t), \phi(t))$ defined in polar coordinates, the curvature is given by

(4.10)
$$\kappa(t) = \frac{(2\dot r^2 \dot\phi + r\dot r\ddot\phi - r\ddot r\dot\phi + r^2\dot\phi^3)}{(\dot r^2 + r^2\dot\phi^2)^{\frac{3}{2}}}.$$

In the $\theta-$parametrization, the configuration of the ring under the deformation (α, w) is given by the curve

(4.11)
$$\begin{aligned} r(\theta) &= a + w(\theta), \\ \phi(\theta) &= \theta + \psi + \alpha(\theta). \end{aligned}$$

Using equation (4.11) in (4.10) to express the curvature of the ring under the deformation (w, α), we find (keeping only terms to first-order in the numerator and denominator)

(4.12)
$$\kappa_{\alpha,w}(\theta) \approx \frac{(-a\frac{\partial^2 w}{\partial \theta^2} + a^2 + 2aw + 3a^2\frac{\partial \alpha}{\partial \theta})}{\left(a^2 + 3a^2 w + 3a^3\frac{\partial \alpha}{\partial \theta}\right)}$$

Writing the curvature in terms of α alone using the inextensibility condition of equation (4.4) yields

(4.13)
$$\kappa_\alpha(\theta) = \frac{\left(\frac{\partial^3 \alpha}{\partial \theta^3} + \frac{\partial \alpha}{\partial \theta} + 1\right)}{a}.$$

From this we have $\kappa_{\alpha \equiv 0}(\theta) = \frac{1}{a}$. Using these results in the potential energy of equation (4.9) yields

(4.14)
$$V = \frac{\beta}{2a} \int_0^{2\pi} \left[\frac{\partial^3 \alpha}{\partial \theta^3} + \frac{\partial \alpha}{\partial \theta}\right]^2 d\theta.$$

In Bryan's original work the potential energy included also a term capturing the work done in stretching the ring and the work done against an attracting force which he introduced to separate the effects of the centrifugal force from the remaining terms. These two terms are related by a simple equation involving the rate of the imposed rotation and at the conclusion of his analysis Bryan chooses the attracting force so as

to cancel the tension, leaving only the work done in bending. Here we take a simpler approach, similar to Rayleigh, and omit those terms at the outset.

The standard Lagrangian function is defined to be the kinetic minus potential energies and is given here by

$$L = \frac{1}{2} \int_0^{2\pi} \left[\left(1 - \frac{\partial \alpha}{\partial \theta} \right)^2 \left(\dot{\psi}^2 + 2\dot{\psi}\dot{\alpha}(\theta) + \dot{\alpha}^2(\theta) \right) + \left(\frac{\partial \dot{\alpha}}{\partial \theta} \right)^2 \right] \sigma a^3 d\theta$$

(4.15)
$$- \frac{\beta}{2a} \int_0^{2\pi} \left[\frac{\partial^3 \alpha}{\partial \theta^3} + \frac{\partial \alpha}{\partial \theta} \right]^2 d\theta.$$

Consider now the following action Φ_g of S^1 on $\mathcal{C}$.

(4.16)
$$\Phi_g(\psi, \alpha) = (\psi + g, \alpha).$$

We have the following lemma.

LEMMA 4.1. $(\mathcal{C}, (\cdot, \cdot), V, S^1)$ *is a simple mechanical system with symmetry where the action of S^1 on $\mathcal{C}$ is given by equation* (4.16). *(For a definition and discussion of simple mechanical systems with symmetry see Section* 4.5 *of* [1].)

Proof. Immediate since both $(\cdot, \cdot)$ and V are invariant under the given action of S^1 on $\mathcal{C}$. $\square$

Since the given action is both free and proper, the reduced space $Q = \mathcal{C}/S^1$, given by

(4.17)
$$Q = \{\alpha : S^1 \to S^1 | \alpha(0) = \alpha(2\pi), \alpha \text{ smooth}\}$$

is also a manifold. To fix notation in relation to the general theory presented in Section 3, we note that $Q = \mathcal{C}/S^1$ is the configuration space for the ring and $S = \mathcal{C}$ is the ambient space in which Q is moved. To slowly rotate the ring we set $\psi = \psi_0 + \Omega t$ (identifying $\psi = 0$ with $\psi = 2\pi$) for some small Ω and some fixed initial offset ψ_0 so that the embedding from Q to S is given by

(4.18)
$$m_t(\alpha(\theta)) = (\psi_0 + \Omega t, \alpha(\theta)).$$

4.1. The nominal dynamics. The nominal dynamics is given by setting $\Omega = 0$ in equation (4.18). Applying this to equation (4.15) yields the nominal Lagrangian.

$$L^{Nom}(\alpha, \dot{\alpha}) = \int_0^{2\pi} \left\{ \frac{\sigma a^3}{2} \left[\left(1 - \frac{\partial \alpha}{\partial \theta} \right)^2 \dot{\alpha}^2 + \left(\frac{\partial \dot{\alpha}}{\partial \theta} \right)^2 \right] \right.$$

(4.19)
$$\left. - \frac{\beta}{2a} \left[\frac{\partial^3 \alpha}{\partial \theta^3} + \frac{\partial \alpha}{\partial \theta} \right]^2 \right\} d\theta$$

The action integral for this Lagrangian is defined to be

(4.20)
$$J(\alpha, \dot{\alpha}) \triangleq \int_a^b L^{Nom}(\alpha, \dot{\alpha})dt.$$

The Euler-Lagrange equations for this system are found by applying Hamilton's principle of critical action (see, e.g. [1]) which states that

(4.21)
$$\delta J(\alpha, \dot{\alpha}) = \delta \int_a^b L^{Nom}(\alpha, \dot{\alpha})dt = 0$$

for all variations among paths $\eta(t)$ in Q with fixed end-points. Applying the variation yields

$$
\delta J(\alpha, \dot\alpha) = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} J(\alpha + \epsilon\eta, \dot\alpha + \epsilon\dot\eta)
$$

$$
= \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \int_a^b \int_0^{2\pi} \left\{ \frac{\sigma a^3}{2}\left[\left(1 - \frac{\partial(\alpha+\epsilon\eta)}{\partial\theta}\right)^2 (\dot\alpha + \epsilon\dot\eta)^2 + \left(\frac{\partial(\dot\alpha+\epsilon\dot\eta)}{\partial\theta}\right)^2\right]\right.
$$

$$
\left. - \frac{\beta}{2a}\left[\frac{\partial^3(\alpha+\epsilon\eta)}{\partial\theta^3} + \frac{\partial(\alpha+\epsilon\eta)}{\partial\theta}\right]^2 \right\} d\theta dt
$$

$$
= \int_a^b \int_0^{2\pi} \left\{ \sigma a^3\left[-\left(1 - \frac{\partial\alpha}{\partial\theta}\right)\dot\alpha^2\frac{\partial\eta}{\partial\theta} + \left(1 - \frac{\partial\alpha}{\partial\theta}\right)^2\dot\alpha\dot\eta + \frac{\partial\dot\alpha}{\partial\theta}\frac{\partial\dot\eta}{\partial\theta}\right]\right.
$$

$$
\left. - \frac{\beta}{a}\left[\frac{\partial^3\alpha}{\partial\theta^3} + \frac{\partial\alpha}{\partial\theta}\right]\left[\frac{\partial^3\eta}{\partial\theta^3} + \frac{\partial\eta}{\partial\theta}\right]\right\} d\theta dt.
$$

Using integration by parts repeatably on the space variable and the fact that for any element $\xi(\cdot) \in Q$ we have $\xi(0) = \xi(2\pi)$, the variation of the action can be rewritten as

$$
\delta J(\alpha, \dot\alpha) = \int_a^b \int_0^{2\pi} \left\{ \sigma a^3\left[2\left(1 - \frac{\partial\alpha}{\partial\theta}\right)\frac{\partial\dot\alpha}{\partial\theta}\dot\alpha\eta - \frac{\partial^2\alpha}{\partial\theta^2}\dot\alpha^2\eta + \left(1 - \frac{\partial\alpha}{\partial\theta}\right)^2\dot\alpha\dot\eta - \frac{\partial^2\dot\alpha}{\partial\theta^2}\dot\eta\right]\right.
$$

$$
\left. + \frac{\beta}{a}\left[\frac{\partial^6\alpha}{\partial\theta^6} + 2\frac{\partial^4\alpha}{\partial\theta^4} + \frac{\partial^2\alpha}{\partial\theta^2}\right]\eta \right\} d\theta dt.
$$

Using integration by parts once again, this time on the time variable, and utilizing the end point condition on the variations η yields

$$
\delta J(\alpha, \dot\alpha) = \int_a^b \int_0^{2\pi} \left\{ \sigma a^3\left[4\left(1 - \frac{\partial\alpha}{\partial\theta}\right)\frac{\partial\dot\alpha}{\partial\theta}\dot\alpha - \frac{\partial^2\alpha}{\partial\theta^2}\dot\alpha^2 - \left(1 - \frac{\partial\alpha}{\partial\theta}\right)^2\ddot\alpha + \frac{\partial^2\ddot\alpha}{\partial\theta^2}\right]\right.
$$

$$
\left. + \frac{\beta}{a}\left[\frac{\partial^6\alpha}{\partial\theta^6} + 2\frac{\partial^4\alpha}{\partial\theta^4} + \frac{\partial^2\alpha}{\partial\theta^2}\right]\right\} \eta d\theta dt.
$$

Since this must equal zero for all variations η we obtain the Euler-Lagrange equations for the nominal system.

$$
0 = \sigma a^3\left[\frac{\partial^2\ddot\alpha}{\partial\theta^2} - \frac{\partial^2\alpha}{\partial\theta^2}\dot\alpha^2 + 4\left(1 - \frac{\partial\alpha}{\partial\theta}\right)\frac{\partial\dot\alpha}{\partial\theta}\dot\alpha - \left(1 - \frac{\partial\alpha}{\partial\theta}\right)^2\ddot\alpha\right]
$$

(4.22)
$$
+ \frac{\beta}{a}\left[\frac{\partial^6\alpha}{\partial\theta^6} + 2\frac{\partial^4\alpha}{\partial\theta^4} + \frac{\partial^2\alpha}{\partial\theta^2}\right].
$$

To simplify this difficult nonlinear partial differential equation we use the assumption that the deformations are small and replace the above equation by its linearization, resulting in the following equation of motion for α.

(4.23)
$$
\sigma a^3\left[\frac{\partial^2\ddot\alpha}{\partial\theta^2} - \ddot\alpha\right] + \frac{\beta}{a}\left[\frac{\partial^6\alpha}{\partial\theta^2} + 2\frac{\partial^4\alpha}{\partial\theta^4} + \frac{\partial^2\alpha}{\partial\theta^2}\right] = 0.
$$

At this stage we state our intention to do all of the calculations associated to the application of the moving systems approach to the vibrating ring problem in a

convenient set of coordinates, namely the coeffecients of $\alpha, \dot{\alpha}$ expressed in a Fourier basis. We first express the nominal dynamics in these coordinates and in the following section do the holonomy calculations in the same coordinates (after truncation of the Fourier series). In these coordinates α has the form

$$(4.24) \qquad \alpha(\theta) = \sum_{k=1}^{\infty} \left[A_k \cos(k\theta) + B_k \sin(k\theta) \right].$$

The deformation $\alpha(\theta)$ is not allowed to contain any global rotations and so the constant coefficient is set to 0. Inserting this expression for α into the equation of motion (4.23) results in the equation

$$0 = \sum_{k=1}^{\infty} \left\{ \sigma a^3 (1 + k^2) \left[\ddot{A}_k \cos(k\theta) + \ddot{B}_k \sin(k\theta) \right] \right.$$

$$(4.25) \qquad \left. + \frac{\beta}{a} (k^6 - 2k^4 + k^2) \left[A_k \cos(k\theta) + B_k \sin(k\theta) \right] \right\}.$$

Collecting terms in $\cos(\theta)$ and $\sin(\theta)$ and setting them separately to zero gives the following set of ordinary differential equations for the Fourier coefficients.

$$(4.26) \qquad \ddot{A}_k = -\frac{\beta}{\sigma a^4} \frac{k^2 (k^2 - 1)^2}{k^2 + 1} A_k \overset{\triangle}{=} -\eta_k^2 A_k,$$

$$(4.27) \qquad \ddot{B}_k = -\frac{\beta}{\sigma a^4} \frac{k^2 (k^2 - 1)^2}{k^2 + 1} A_k \overset{\triangle}{=} -\eta_k^2 B_k$$

which defines the frequencies η_k. This result is in agreement with a derivation of Rayleigh [23] and defines for each k a pair of uncoupled oscillators with common frequency η_k. The solution to this system is given by

$$(4.28) \qquad A_k(t) = \widehat{A}_k \cos(\eta_k t) + \frac{\widehat{\dot{A}}_k}{\eta_k} \sin(\eta_k t),$$

$$(4.29) \qquad B_k(t) = \widehat{B}_k \cos(\eta_k t) + \frac{\widehat{\dot{B}}_k}{\eta_k} \sin(\eta_k t)$$

where $\widehat{A}_k, \widehat{\dot{A}}_k, \widehat{B}_k$, and $\widehat{\dot{B}}_k$ are given by initial conditions. The Hannay-Berry phase is defined as the holonomy on a trivial bundle involving the cotangent space of the system. It will prove useful, then, to have the time evolution of the conjugate momenta for the nominal system. By inserting the Fourier expansion for α into the nominal Lagrangian, equation (4.15), and applying the Legendre transform we obtain

$$(4.30) \qquad p_{A_k} = \frac{\partial L}{\partial \dot{A}_k} = (1 + k^2) \sigma a^3 \pi \dot{A}_k,$$

$$(4.31) \qquad p_{B_k} = \frac{\partial L}{\partial \dot{B}_k} = (1 + k^2) \sigma a^3 \pi \dot{B}_k.$$

Thus the solution to the nominal system expressed on the cotangent bundle is given by

$$(4.32) \qquad A_k(t) = \widehat{A}_k \cos(\eta_k t) + \frac{\widehat{p}_{A_k}}{(1 + k^2) \sigma a^3 \pi \eta_k} \sin(\eta_k t),$$

(4.33) $$p_{A_k}(t) = -(1+k^2)\sigma a^3 \pi \eta_k \widehat{A}_k \sin(\eta_k t) + \widehat{p}_{A_k} \cos(\eta_k t,)$$

(4.34) $$B_k(t) = \widehat{B}_k \cos(\eta_k t) + \frac{\widehat{p}_{B_k}}{(1+k^2)\sigma a^3 \pi \eta_k} \sin(\eta_k t),$$

(4.35) $$p_{B_k}(t) = -(1+k^2)\sigma a^3 \pi \eta_k \widehat{B}_k \sin(\eta_k t) + \widehat{p}_{B_k} \cos(\eta_k t).$$

4.2. The Hannay-Berry phase. From equation (4.18), the velocity vector of the motion in S is

(4.36) $$\frac{d}{dt}\left(m_t(\alpha(\theta))\right) = (0, \dot\alpha(\theta)) + (\Omega, 0).$$

From this we see that the vector field corresponding to the imposed motion is

(4.37) $$\mathcal{Z}(m_t(\alpha(\theta))) = \begin{pmatrix} \Omega \\ 0 \end{pmatrix}.$$

For ease of notation define $\mathcal{Z} \overset{\triangle}{=} \mathcal{Z}_t(m_t(\alpha(\theta)))$. The projection of this tangent vector to $T_{m_t(q)}m_t(Q)$ with respect to the metric of S is given by $\mathcal{Z}^T = \mathcal{Z} - \mathcal{Z}^\perp$ where $(\mathcal{Z}^\perp, X) = 0 \ \forall X \in T_{m_t(q)}m_t(Q)$. From equation (4.6), any vector $X \in T_{m_t(q)}m_t(Q)$ has the form

(4.38) $$X = \begin{pmatrix} 0 \\ Y \end{pmatrix}$$

where $Y \in T_q Q$. Then, from equation (4.37) and the fact that $\mathcal{Z}^T \in T_{m_t(q)}m_t(Q)$, we can write $\mathcal{Z}^T = \mathcal{Z} - \mathcal{Z}^\perp$ as

(4.39) $$\begin{pmatrix} 0 \\ Y_{\mathcal{Z}^T} \end{pmatrix} = \begin{pmatrix} \Omega \\ 0 \end{pmatrix} - \begin{pmatrix} \mathcal{Z}^{\perp_1} \\ \mathcal{Z}^{\perp_2} \end{pmatrix}$$

for some $Y_{\mathcal{Z}^T}, \mathcal{Z}^{\perp_2} \in T_q(Q)$ and $\mathcal{Z}^{\perp_1} \in T_\psi S^1$. From equation (4.39), $\mathcal{Z}^{\perp_1} = \Omega$. Applying equation (4.8), the orthogonality condition states

(4.40)
$$0 = ((\Omega, \mathcal{Z}^{\perp_2}), (0, Y))$$
$$= \int_0^{2\pi} \left[\left(1 - \frac{\partial\alpha}{\partial\theta}\right)^2 (\Omega Y + \mathcal{Z}^{\perp_2} Y) + \frac{\partial\mathcal{Z}^{\perp_2}}{\partial\theta}\frac{\partial Y}{\partial\theta} \right] \sigma a^3 d\theta$$

In what follows, we express the orthogonality condition of equation (4.40) in the Fourier basis and thus derive an explicit formula for $\mathcal{Z}^{\perp_2}$ (see (4.51) and (4.52) below). Using the Fourier series representation for α, a tangent vector $Y \in T_q Q$ has the form

(4.41) $$Y = \sum_{k=1}^{\infty} [Y_{A_k}\cos(k\theta) + Y_{B_k}\sin(k\theta)]$$

Adopting the notation $c(\theta) = \cos(\theta)$ and $s(\theta) = \sin(\theta)$, the orthogonality condition is given by

$$0 = \int_0^{2\pi} \left[\left(1 - \sum_{k=1}^{\infty} k\left[B_k c(k\theta) - A_k s(k\theta)\right]\right)^2 \left(\Omega \sum_{k=1}^{\infty} [Y_{A_k}c(k\theta) + Y_{B_k}s(k\theta)] \right.\right.$$
$$+ \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\left[\mathcal{Z}_{A_k}^{\perp_2}c(k\theta) + \mathcal{Z}_{B_k}^{\perp_2}s(k\theta)\right][Y_{A_l}c(l\theta) + Y_{B_l}s(l\theta)]\bigg)$$

(4.42) $$+ \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} kl\left(\mathcal{Z}_{B_k}^{\perp_2}c(k\theta) - \mathcal{Z}_{A_k}^{\perp_2}s(k\theta)\right)(Y_{B_l}c(l\theta) - Y_{A_l}s(l\theta))\Bigg] \sigma a^3 d\theta.$$

Using the following identities

(4.43)
$$\int_0^{2\pi} c(k\theta)c(l\theta)d\theta = \int_0^{2\pi} s(k\theta)s(l\theta)d\theta = \pi\delta_{kl},$$

(4.44)
$$\int_0^{2\pi} c(k\theta)c(l\theta)c(m\theta)c(n\theta)d\theta = \frac{3\pi}{4}\delta_{klmn},$$

(4.45)
$$\int_0^{2\pi} s(k\theta)s(l\theta)s(m\theta)s(n\theta)d\theta = \frac{3\pi}{4}\delta_{klmn},$$

(4.46)
$$\int_0^{2\pi} c(k\theta)c(l\theta)s(m\theta)s(n\theta)d\theta = \frac{\pi}{4}\delta_{klmn}$$

where

(4.47)
$$\delta_{kl} = \begin{cases} 1 & \text{if } k = l \\ 0 & \text{otherwise} \end{cases} \qquad \delta_{klmn} = \begin{cases} 1 & \text{if } k = l = m = n \\ 0 & \text{otherwise} \end{cases}$$

and the fact that all other combinations of sin and cos appearing in equation (4.42) integrate to 0 over $[0, 2\pi]$, the orthogonality condition can be reduced to

$$0 = \sigma a^3 \pi \sum_{k=1}^{\infty} \left[\left(\mathcal{Z}_{A_k}^{\perp_2} + \frac{k^2}{4} A_k^2 \mathcal{Z}_{A_k}^{\perp_2} + \frac{3k^2}{4} B_k^2 \mathcal{Z}_{A_k}^{\perp_2} - \frac{k^2}{2} A_k B_k \mathcal{Z}_{B_k}^{\perp_2} \right. \right.$$

$$\left. - 2\Omega k B_k + k^2 \mathcal{Z}_{A_k}^{\perp_2} \right) Y_{A_k}$$

$$+ \left(\mathcal{Z}_{B_k}^{\perp_2} + \frac{3k^2}{4} A_k^2 \mathcal{Z}_{B_k}^{\perp_2} + \frac{k^2}{4} B_k^2 \mathcal{Z}_{B_k}^{\perp_2} - \frac{k^2}{2} A_k B_k \mathcal{Z}_{A_k}^{\perp_2} \right.$$

(4.48)
$$\left. \left. + 2\Omega k A_k + k^2 \mathcal{Z}_{B_k}^{\perp_2} \right) Y_{B_k} \right].$$

This holds for every $Y \in T_q Q$ and so for every k we have

(4.49) $\mathcal{Z}_{A_k}^{\perp_2} + \dfrac{k^2}{4} A_k^2 \mathcal{Z}_{A_k}^{\perp_2} + \dfrac{3k^2}{4} B_k^2 \mathcal{Z}_{A_k}^{\perp_2} - \dfrac{k^2}{2} A_k B_k \mathcal{Z}_{B_k}^{\perp_2} - 2\Omega k B_k + k^2 \mathcal{Z}_{A_k}^{\perp_2} = 0$

(4.50) $\mathcal{Z}_{B_k}^{\perp_2} + \dfrac{3k^2}{4} A_k^2 \mathcal{Z}_{B_k}^{\perp_2} + \dfrac{k^2}{4} B_k^2 \mathcal{Z}_{B_k}^{\perp_2} - \dfrac{k^2}{2} A_k B_k \mathcal{Z}_{A_k}^{\perp_2} + 2\Omega k A_k + k^2 \mathcal{Z}_{B_k}^{\perp_2} = 0.$

Solving these coupled equations for $\mathcal{Z}_{A_k}^{\perp_2}$ and $\mathcal{Z}_{B_k}^{\perp_2}$ yields

(4.51)
$$\mathcal{Z}_{A_k}^{\perp_2} = \Omega \left[\frac{2k(1 + k^2 + \frac{3k^2}{4}A_k^2 + \frac{k^2}{4}B_k^2)B_k - k^3 A_k^2 B_k}{D_k(A_k, B_k)} \right],$$

(4.52)
$$\mathcal{Z}_{B_k}^{\perp_2} = -\Omega \left[\frac{2k(1 + k^2 + \frac{k^2}{4}A_k^2 + \frac{3k^2}{4}B_k^2)A_k - k^3 A_k B_k^2}{D_k(A_k, B_k)} \right]$$

where

$$D_k(A_k, B_k) = (1 + k^2 + \frac{k^2}{4}A_k^2 + \frac{3k^2}{4}B_k^2)(1 + k^2 + \frac{3k^2}{4}A_k^2 + \frac{k^2}{4}B_k^2)$$

(4.53)
$$- \frac{k^4}{4} A_k^2 B_k^2$$

and so, representing the tangent vector by its coefficients at each k,

$$(4.54) \qquad \mathcal{Z}^{\perp} = \begin{pmatrix} \Omega \\ \left\{ \Omega \left[\frac{2k(1+k^2+\frac{3k^2}{4}A_k^2+\frac{k^2}{4}B_k^2)B_k-k^3A_k^2B_k}{D_k(A_k,B_k)} \right] \right\}_{k=1}^{\infty} \\ \left\{ -\Omega \left[\frac{2k(1+k^2+\frac{k^2}{4}A_k^2+\frac{3k^2}{4}B_k^2)A_k-k^3A_kB_k^2}{D_k(A_k,B_k)} \right] \right\}_{k=1}^{\infty} \end{pmatrix}.$$

Inserting equation (4.54) into equation (4.39) gives the projection of the tangent vector of the imposed motion onto $T_{m_t(q)}m_t(Q)$ to be

$$(4.55) \qquad \mathcal{Z}^{T} = \begin{pmatrix} 0 \\ \left\{ -\Omega \left[\frac{2k(1+k^2+\frac{3k^2}{4}A_k^2+\frac{k^2}{4}B_k^2)B_k-k^3A_k^2B_k}{D_k(A_k,B_k)} \right] \right\}_{k=1}^{\infty} \\ \left\{ \Omega \left[\frac{2k(1+k^2+\frac{k^2}{4}A_k^2+\frac{3k^2}{4}B_k^2)A_k-k^3A_kB_k^2}{D_k(A_k,B_k)} \right] \right\}_{k=1}^{\infty} \end{pmatrix}.$$

The pull-back of $\mathcal{Z}^{T}$ to T_qQ by $[Tm]^{-1}$ is given by

$$(4.56)\ Z(q) \triangleq [Tm]^{-1}\mathcal{Z}^{T} = \begin{pmatrix} \left\{ -\Omega \left[\frac{2k(1+k^2+\frac{3k^2}{4}A_k^2+\frac{k^2}{4}B_k^2)B_k-k^3A_k^2B_k}{D_k(A_k,B_k)} \right] \right\}_{k=1}^{\infty} \\ \left\{ \Omega \left[\frac{2k(1+k^2+\frac{k^2}{4}A_k^2+\frac{3k^2}{4}B_k^2)A_k-k^3A_kB_k^2}{D_k(A_k,B_k)} \right] \right\}_{k=1}^{\infty} \end{pmatrix}$$

where $Z(q)$ is defined for ease of notation. Recalling that the deformations are assumed to be small, the above expression is expanded in a Taylor series about $A_k = B_k = 0 \ \forall \ k$ and only the first order terms kept. This yields

$$(4.57) \qquad Z(q) = \begin{pmatrix} \left\{ \frac{-2\Omega k}{1+k^2}B_k \right\}_{k=1}^{\infty} \\ \left\{ \frac{2\Omega k}{1+k^2}A_k \right\}_{k=1}^{\infty} \end{pmatrix}.$$

Let $q_k = (A_k, B_k)$ so that the coordinates on Q are $q = \{q_k\}_{k=1}^{\infty}$. The conjugate momenta can then be written $p_k = (p_{A_k}, p_{B_k})$. Define

$$(4.58) \qquad Z_k(q_k) \triangleq \frac{2k\Omega}{1+k^2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A_k \\ B_k \end{pmatrix} = \frac{2k\Omega}{1+k^2}S(1)q_k$$

which also defines the skew symmetric matrix $S(1)$. With this definition the projected vector can be expressed as $Z(q) = \{Z_k(q_k)\}_{k=1}^{\infty}$. To avoid technical difficulties we assume that only N Fourier modes are active where N is some positive, finite number. Using this assumption, the function $\mathcal{P}(Z)$ defining the horizontal lift relative to the induced Cartan connection (as in equation (3.3)) is given by

$$(4.59) \qquad \mathcal{P}(Z)(q,p) = p \cdot Z(q) = \sum_{k=1}^{N} \left[\frac{2k\Omega}{1+k^2} \right] p_k \cdot S(1)q_k.$$

Define $Q_k = \{(A_k, B_k) \in \mathbb{R}^2\}$ so that $Q = \cup_{k=1}^N Q_k$. The configuration space is then the Cartesian product of N copies of $\mathbb{R}^2$ and each coordinate q_k and conjugate momenta p_k can be identified with a vector in $\mathbb{R}^2$. Extend these vectors to $\mathbb{R}^3$ by letting the third coordinate of each be zero. Let $\widehat{x}_{3_k}$ be a unit vector at the origin of the kth copy of $\mathbb{R}^2$ along this third direction. With these identifications, in equation (4.59) we replace $p_k \cdot S(1)q_k$ by $(q_k \times p_k) \cdot \widehat{x}_{3_k}$. Define

$$(4.60) \qquad\qquad I_k \overset{\triangle}{=} (q_k \times p_k) \cdot \widehat{x}_{3_k}$$

so that

$$(4.61) \qquad\qquad \mathcal{P}(Z)(q,p) = \sum_{k=1}^N \frac{2k}{1+k^2} \Omega I_k.$$

Let $\mathcal{F}$ be the subset of C^∞ functions on T^*Q defined by

$$(4.62) \quad \mathcal{F} = \left\{ f(q,p) | f(q,p) = \sum_{k=1}^N a_k f_k(A_k, B_k, p_{A_k}, p_{B_k}), a_k \in \mathbb{R}, f_k smooth \right\}.$$

Since the time solution to each Fourier mode for the nominal system is periodic, we can define an average on $\mathcal{F}$ with respect to the flow by

$$(4.63) \qquad\qquad \langle f \rangle = \sum_{k=1}^N a_k \langle f_k \rangle_k$$

where

$$(4.64) \qquad\qquad \langle f_k \rangle_k = \frac{\eta_k}{2\pi} \int_0^{\frac{2\pi}{\eta_k}} \left(\phi_t^k \right)^* f_k dt$$

where ϕ_t^k is the flow corresponding to the kth Fourier mode of the nominal system. From equations (4.60) and (4.61) it is clear that $P(Z) \in \mathcal{F}$. We have the following useful lemma.

LEMMA 4.2. I_k is constant along the trajectories of the nominal system.

Proof.

$$\begin{aligned}
\frac{dI_k}{dt} &= \frac{d}{dt}(q_k \times p_k) \cdot \widehat{x}_{3_k} \\
&= \frac{d}{dt}(p_{B_k} A_k - p_{A_k} B_k) \\
&= \dot{p}_{B_k} A_k + p_{B_k} \dot{A}_k - \dot{p}_{A_k} B_k - p_{A_k} \dot{B}_k \\
&= (1+k^2)\sigma a^3 \pi \eta_k \left(\ddot{B}_k A_k + \dot{B}_k \dot{A}_k - \ddot{A}_k B_k - \dot{A}_k \dot{B}_k \right) \\
&= (1+k^2)\sigma a^3 \pi \eta_k \left(\ddot{B}_k A_k - \ddot{A}_k B_k \right)
\end{aligned}$$

where in the second to last step we have used the definition of the conjugate momenta for the nominal system in equations (4.30,4.31). From equations (4.26,4.27), along the trajectories of the nominal system we have

$$(4.65) \qquad\qquad \left(\ddot{B}_k A_k - \ddot{A}_k B_k \right) = -\eta_k^2 \left(A_k B_k - A_k B_k \right) = 0.$$

⬜

The average of $\mathcal{P}(Z)$ over the nominal dynamics is then

$$(4.66) \qquad \langle \mathcal{P}(Z) \rangle (q, p) = \sum_{k=1}^{N} \frac{2k}{1+k^2} \Omega \langle I_k \rangle_k = \sum_{k=1}^{N} \frac{2k}{1+k^2} \Omega I_k.$$

Noticing that this function depends only the coordinates (q_k, p_k) through I_k, we move to the coordinates for the averaged dynamics in phase space defined by $(\phi_k, I_k, \rho_k, p_{\rho_k})$ where $\rho_k = \langle (A_k^2 + B_k^2) \rangle^{\frac{1}{2}}$. Here ϕ_k is conjugate to I_k and p_{ρ_k} is conjugate to ρ_k.

With these coordinates the horizontal lift of Ω relative to the Cartan-Hannay-Berry connection (as in equation (3.6)) is given by

$$(4.67) \qquad \left(-X_{\langle \mathcal{P}(Z(q)) \rangle}, \Omega \right) = \left(\left\{ -\frac{2k}{1+k^2} \Omega \frac{\partial}{\partial \phi_k} \right\}_{k=1}^{N}, \Omega \right).$$

The geometric phase is neatly split into a phase change in each ϕ_k independently. If the loop in M is parametrized by $t \in [0, T]$ where $T = \frac{2\pi}{\Omega}$ is the time to complete one full revolution of the ring, then

$$(4.68) \qquad \Delta \phi_k = -\int_0^T \frac{2k}{1+k^2} \Omega dt = -\frac{2k}{1+k^2} \Omega T = -2\pi \left[\frac{2k}{1+k^2} \right].$$

After one full rotation of the ring each vector q_k has been rotated by the angle $\Delta \phi_k$. In practice one expects to use only one mode under a resonant drive (as in [21, 22]).

Note that we perform an average with respect to the flow of the nominal dynamics on a special class of functions as in equation (4.62). This agrees with a group (here S^1) average as in Section 3 (Definition 3.3) when we restrict the nominal dynamics to a single mode.

5. A comparison with the results of Bryan . In [8] Bryan uses classical variational techniques to derive the equations of motion for a thin ring of radius a undergoing a steady rotation about its central axis with angular velocity Ω. His analysis uses two polar coordinate systems, the first fixed in space and the second rotating with angular velocity Ω. If in the undeformed state the coordinate systems are given by (a, ϕ) and (a, θ) we have

$$(5.1) \qquad \phi = \theta + \Omega t$$

and θ is constant for any particle of the ring. Let the tangential and radial displacements of a particle of the ring be given by v and w respectively so that the new polar coordinates are $(a + w, \phi + v/a)$ and $(a + w, \theta + v/a)$ in the two systems. The deformations v, w are assumed to be small. The assumption of inextensibility yields

$$(5.2) \qquad w = -\frac{\partial v}{\partial \theta}$$

as before. As discussed in the previous section, Bryan includes work done against the tension, $\mathcal{T}$, to stretch the ring and against an attractive force, μ. To match his derivation with the model we have chosen, we set these terms to zero. Taking variations on the total energy and setting them to zero, we find the following equation of motion

$$(5.3) \qquad 0 = \ddot{v} - \frac{\partial^2 \ddot{v}}{\partial \theta^2} - 4\Omega \frac{\partial \dot{v}}{\partial \theta} + \Omega^2 \frac{\partial^2 v}{\partial \theta^2} - \frac{\beta}{\sigma a^4} \frac{\partial^2}{\partial \theta^2} \left(\frac{\partial^2}{\partial \theta^2} + 1 \right)^2 v.$$

Bryan then assumes that the deformations are of the form

$$(5.4) \qquad v = cos(k\theta + pt).$$

Inserting this into the equation of motion yields the following two solutions.

$$(5.5) \qquad v = Acos\left(k\theta + \frac{2k}{1+k^2}\Omega t + \bar{\omega}_k t\right),$$

$$(5.6) \qquad v = Acos\left(k\theta + \frac{2k}{1+k^2}\Omega t - \bar{\omega}_k t\right)$$

where

$$\bar{\omega}_k^2 = \frac{\beta}{\sigma a^4}\frac{k^2(k^2-1)^2}{1+k^2} - \frac{\Omega^2 k^2(k^2-3)}{(1+k^2)^2}$$

$$(5.7) \qquad\qquad = \eta_k^2 - \frac{\Omega^2 k^2(k^2-3)}{(1+k^2)^2}.$$

Notice that by retaining the terms in Ω^2 there is a slight decrease in the frequency of vibration from η_k. This can be understood as a "softening" of the material and corresponds to the spurious softening that occurs in the theory of rotating rods if the models are linearized prematurely, that is before the effects of external rotation are considered. The geometrically exact theory handles this issue properly and we believe that extending the model of the ring using this theory will prove useful. (We note also that if the attractive force μ introduced by Bryan is kept in the equations of motion and set equal to Ω^2 then the resulting system shows an increase in the frequency of vibration due to the imposed rotation. See [13] for comments on similar ad hoc methods in the theory of rotating beams.)

Assuming the amplitude A in the two solutions is the same, the final solution in the fixed frame is given by

$$(5.8) \qquad v = 2Acos(\bar{\omega}_k t)cos\left(k\left[\phi - \frac{k^2-1}{k^2+1}\Omega t\right]\right).$$

Bryan then recognizes that this corresponds to an oscillation with $2k$ nodes where the position of the nodes precess in retrograde around the ring with angular velocity

$$(5.9) \qquad \frac{k^2-1}{k^2+1}\Omega.$$

If we write this in the rotating system we have

$$(5.10) \qquad \left(\frac{k^2-1}{k^2+1}-1\right)\Omega = -\frac{2}{k^2+1}\Omega$$

and after one rotation the nodes have precessed in the moving frame by

$$(5.11) \qquad -2\pi\left(\frac{2}{1+k^2}\right).$$

To compare this to our results in Section 4 we must first restrict our solution to a single mode so that

$$(5.12) \qquad \alpha(\theta,t) = A_k(t)\cos(k\theta) + B_k(t)\sin(k\theta).$$

When (A_k, B_k) is viewed as a vector in $\mathbb{R}^2$, the effect of the geometric phase is seen to be a rotation of this vector about the origin where the counter-clockwise direction is defined to be positive. Using equation (4.68), the rotated vector at the end of one revolution of the ring is given by

$$
\begin{pmatrix} A_k(T) \\ B_k(T) \end{pmatrix} = \begin{pmatrix} \cos\left(-2\pi\frac{2k}{1+k^2}\right) & -\sin\left(-2\pi\frac{2k}{1+k^2}\right) \\ \sin\left(-2\pi\frac{2k}{1+k^2}\right) & \cos\left(-2\pi\frac{2k}{1+k^2}\right) \end{pmatrix} \begin{pmatrix} A_k(0) \\ B_k(0) \end{pmatrix}
$$

(5.13)
$$
= \begin{pmatrix} A_k(0)\cos\left(-2\pi\frac{2k}{1+k^2}\right) - B_k(0)\sin\left(-2\pi\frac{2k}{1+k^2}\right) \\ A_k(0)\sin\left(-2\pi\frac{2k}{1+k^2}\right) + B_k(0)\cos\left(-2\pi\frac{2k}{1+k^2}\right) \end{pmatrix}.
$$

Inserting this into equation (5.12) and simplifying we get

$$
\alpha(\theta, T) = A_k(0)\cos\left(k\left[\theta + 2\pi\frac{2}{1+k^2}\right]\right)
$$

(5.14)
$$
+ B_k(0)\sin\left(k\left[\theta + 2\pi\frac{2}{1+k^2}\right]\right)
$$

which is course the same solution with the nodes rotated by $-2\pi\left[\frac{2}{1+k^2}\right]$, agreeing with Bryan.

6. Nonlinear corrections . We now turn to an investigation of corrections to the geometric phase based on the nonlinear terms in the vector field $Z(q)$ given in equation (4.56). It is worth noting that these arise due to the configuration-dependent quadratic form defining the kinetic energy. We proceed by keeping higher-order terms in the Taylor expansion of $Z(q)$. The second-order terms in this expansion can be shown to be zero. To third order the vector field is

(6.1)
$$
Z(q) = \begin{pmatrix} \left\{-\Omega\left[\frac{2kB_k}{1+k^2} - \frac{k^2(2+k)A_k^2 B_k}{(1+k^2)^2} - \frac{9k^3 B_k^3}{(1+k^2)^2}\right]\right\}_{k=1}^N \\ \left\{\Omega\left[\frac{2kA_k}{1+k^2} - \frac{k^2(2+k)A_k B_k^2}{(1+k^2)^2} - \frac{9k^3 A_k^3}{(1+k^2)^2}\right]\right\}_{k=1}^N \end{pmatrix}.
$$

Define the matrix $U(q_k)$ by

(6.2)
$$
U(q_k) = \begin{pmatrix} A_k^2 & 0 \\ 0 & B_k^2 \end{pmatrix}.
$$

With this, equation (6.1) can be written as

$$
Z(q) = \left\{\frac{2k\Omega}{1+k^2}S(1)q_k - \frac{k^2(2+k)\Omega}{(1+k^2)^2}U(q_k)S(1)q_k\right.
$$

(6.3)
$$
\left. - \frac{9k^3\Omega}{(1+k^2)^2}S(1)U(q_k)q_k\right\}_{k=1}^N
$$

and the function defining the Hannay-Berry phase is given by

$$
\mathcal{P}(Z)(q,p) = \sum_{k=1}^N \left[\frac{2k\Omega}{1+k^2}p_k \cdot S(1)q_k - \frac{k^2(2+k)\Omega}{(1+k^2)^2}p_k \cdot U(q_k)S(1)q_k\right.
$$

$$-\frac{9k^3\Omega}{(1+k^2)^2}p_k \cdot S(1)U(q_k)q_k\Bigg]$$

$$=\sum_{k=1}^{N}\Bigg[\frac{2k\Omega}{1+k^2}I_k-\frac{k^2(2+k)\Omega}{(1+k^2)^2}p_k \cdot U(q_k)S(1)q_k$$

(6.4)
$$-\frac{9k^3\Omega}{(1+k^2)^2}p_k \cdot S(1)U(q_k)q_k\Bigg]$$

where in the second step the definition of I_k from equation (4.60) has been used. To determine the Hannay-Berry phase we need to find the average of equation (6.4) over the nominal dynamics. In Section 4 we have shown that the first term in square brackets in the sum in equation (6.4) is a constant along trajectories of the nominal system. The second and third terms, however, are not constant and their averages need to be explicitly calculated. For the second term the average of the kth element of the sum is given by

(6.5)
$$\langle p_k \cdot U(q_k)S(1)q_k\rangle_k = \frac{\eta_k}{2\pi}\int_0^{\frac{2\pi}{\eta_k}}\Big[-p_{A_k}(t)A_k^2(t)B_k(t)$$
$$+p_{B_k}(t)A_k(t)B_k^2(t)\Big]\,dt.$$

Using the solution to the nominal system given in equations (4.32–4.35) and making a change of variables in the integration, this becomes

$$\langle p_k \cdot U(q_k)S(1)q_k\rangle_k = \frac{1}{2\pi}\int_0^{2\pi}\Big\{\big[(1+k^2)\sigma a^3\pi\eta_k A_k\sin(t)-p_{A_k}\cos(t)\big]$$

$$\cdot\Big[A_k\cos(t)+\frac{p_{A_k}\sin(t)}{(1+k^2)\sigma a^3\pi\eta_k}\Big]^2\Big[B_k\cos(t)+\frac{p_{B_k}\sin(t)}{(1+k^2)\sigma a^3\pi\eta_k}\Big]$$

$$+\big[-(1+k^2)\sigma a^3\pi\eta_k B_k\sin(t)+p_{B_k}\cos(t)\big]$$

(6.6)
$$\cdot\Big[A_k\cos(t)+\frac{p_{A_k}\sin(t)}{(1+k^2)\sigma a^3\pi\eta_k}\Big]\Big[B_k\cos(t)+\frac{p_{B_k}\sin(t)}{(1+k^2)\sigma a^3\pi\eta_k}\Big]^2\Big\}\,dt$$

where, through a standard abuse of notation in averaging, the hats have been dropped on the initial conditions. The integration identities in equations (4.43 – 4.46) can be used to write the above expression as

(6.7)
$$\langle p_k \cdot U(q_k)S(1)q_k\rangle_k = \frac{1}{8}\Big[\big(A_k^3 p_{B_k}-B_k^3 p_{A_k}-A_k^2 B_k p_{A_k}+A_k B_k^2 p_{B_k}\big)$$
$$+\Big(\frac{A_k p_{A_k}^2 p_{B_k}-B_k p_{A_k}p_{B_k}^2-B_k p_{A_k}^3+A_k p_{B_k}^3}{((1+k^2)\sigma a^3\pi\eta_k)^2}\Big)\Big].$$

Now

(6.8)
$$A_k^3 p_{B_k}-B_k^3 p_{A_k}-A_k^2 B_k p_{A_k}+A_k B_k^2 p_{B_k}=(A_k p_{B_k}-B_k p_{A_k})(A_k^2+B_k^2)$$
$$=I_k(A_k^2+B_k^2)$$

and

(6.9)
$$A_k p_{A_k}^2 p_{B_k}-B_k p_{A_k}p_{B_k}^2-B_k p_{A_k}^3+A_k p_{B_k}^3=(A_k p_{B_k}-B_k p_{A_k})(p_{A_k}^2+p_{B_k}^2)$$
$$=I_k(p_{A_k}^2+p_{B_k}^2)$$

and so equation (6.7) takes the form

$$(6.10) \qquad \langle p_k \cdot U(q_k)S(1)q_k \rangle_k = \frac{I_k}{8}\left(A_k^2 + B_k^2 + \frac{p_{A_k}^2 + p_{B_k}^2}{((1+k^2)\sigma a^3 \pi \eta_k)^2}\right).$$

Following the same procedure, the average of the kth element in the sum of the third term is

$$\langle p_k \cdot S(1)U(q_k)q_k \rangle_k = \frac{\eta_k}{2\pi}\int_0^{2\pi}(p_{B_k}(t)A_k^3(t) - p_{A_k}(t)B_k^3(t))dt$$

$$= \frac{3}{8}\Big[(A_kB_kp_{B_k} - A_k^2B_kp_{A_k} + A_k^3p_{B_k} - B_k^3p_{A_k})$$

$$+ \frac{A_kp_{B_k}^3 - B_kp_{A_k}^3 + A_kp_{A_k}^2p_{B_k} - B_kp_{A_k}p_{B_k}^2}{((1+k^2)\sigma a^3\pi\eta_k)^2}\Big]$$

$$(6.11) \qquad = \frac{3I_k}{8}\left(A_k^2 + B_k^2 + \frac{p_{A_k}^2 + p_{B_k}^2}{((1+k^2)\sigma a^3\pi\eta_k)^2}\right).$$

Using equations (6.10,6.11), the average of $\mathcal{P}(Z)$ is given by

$$\langle \mathcal{P}(Z)\langle(q,p) = \sum_{k=1}^N \Omega I_k \Big[\frac{2k}{1+k^2}$$

$$(6.12) \qquad - \frac{k^2 + 14k^3}{2(1+k^2)^2}\left(\frac{A_k^2 + B_k^2 + \frac{p_{A_k}^2 + p_{B_k}^2}{((1+k^2)\sigma a^3\pi\eta_k)^2}}{2}\right)\Big].$$

From equations (4.32,4.34) we see that the term in parentheses is the average of $(A_k^2 + B_k^2)$ over the nominal dynamics. We move to the averaged coordinates $(\phi_k, I_k, \rho_k, p_{\rho_k})$ as in the comments following equation (4.66). In these coordinates, the average of $\mathcal{P}(Z)$ has the form

$$(6.13) \qquad \langle \mathcal{P}(Z)\langle(q,p) = \sum_{k=1}^N \Omega I_k \Big[\frac{2k}{1+k^2} - \frac{k^2 + 14k^3}{2(1+k^2)^2}\rho_k^2\Big].$$

The lift to third-order of Ω with respect to the Cartan-Hannay-Berry connection is given by

$$(-X_{\langle\mathcal{P}(Z)\rangle}, \Omega) = \Big(\Big\{-\Omega\Big[\frac{2k}{1+k^2} - \frac{k^2 + 14k^3}{2(1+k^2)^2}\rho_k^2\Big]\frac{\partial}{\partial\phi_k}$$

$$(6.14) \qquad -\Omega I_k\rho_k\Big[\frac{k^2+14k^3}{(1+k^2)^2}\Big]\frac{\partial}{\partial p_{\rho_k}}\Big\}_{k=1}^N, \Omega\Big).$$

From equation (6.14) we see that both I_k and ρ_k are constant. Thus

$$\Delta\phi_k = -\int_0^T \Omega\Big[\frac{2k}{1+k^2} - \frac{k^2+14k^3}{2(1+k^2)^2}\rho_k^2\Big]dt$$

$$(6.15) \qquad = -2\pi\Big[\frac{2k}{1+k^2} - \frac{k^2+14k^3}{2(1+k^2)^2}\rho_k^2\Big].$$

Notice that the third-order terms act to reduce the rate of nodal rotation and thus the sensitivity of a vibrating ring gyroscope cannot be increased by increasing the amplitude of vibration and using the nonlinear effects.

In contrast to the earlier calculation where we kept in Z only the terms linear in configuration variables, in the present nonlinear setting the imposed rotation causes not only a precession of the nodes of vibration but also a drift in the momentum conjugate to ρ_k. In practical devices the ring is driven into a single mode of oscillation and the imposed rotation sensed by measuring the drift rate of the nodal points of the vibration. Thus the effect of the second term in equation (6.14) will be compensated for by the drive electronics.

It is interesting to ask how large the nonlinear effect on the drift rate of the nodal points of the vibrations is in a typical device. The micromachined ring gyroscope of Putty and Najafi [22] utilizes a ring of radius $a = 500$ μm placed into elliptical vibration so that $k = 2$ with a radial deformation amplitude of 0.15 μm. From equations (4.4,4.24), the radial deformation for this ring is

$$(6.16) \qquad w(\theta) = 2a \left[A_2 \sin(2\theta) - B_2 \cos(2\theta)\right].$$

Let $t = 0$ be the time at which the maximum radial deformation is attained and let $\theta = 0$ be the location on the ring of the maximum radial deformation at time $t = 0$. From these definitions we have the initial conditions $\widehat{A}_2 = \widehat{p}_{A_2} = \widehat{p}_{B_2} = 0$ and $\widehat{B}_2 = -\frac{0.15\mu m}{2(500\mu m)}$. Inserting these values into equation (6.15) yields

$$(6.17) \qquad \Delta\phi_2 = -2\pi \left[\frac{4}{5} - 2.61 \times 10^{-8}\right].$$

For the normal operation of this device, then, the nonlinear effects are seven orders of magnitude smaller than the first-order terms.

7. Conclusions and future work . In this paper we have applied the moving systems approach developed by Marsden, Montgomery, and Ratiu to the rotating, vibrating ring and showed that the precession of the nodes with respect to the ring can be understood as the Hannay-Berry phase. We have also showed that under the same linearizing assumptions, our result matches that of Bryan's original analysis. Using the inherently nonlinear nature of the moving systems approach we then calculated the effect of the imposed rotation on the nodal precession to third-order. These higher-order terms serve to reduce the sensitivity of the device and we therefore conclude that the best performance of these devices is obtained by operating them in the linear regime.

There is an underlying assumption that the rate of rotation of the ring is slow with respect to the frequency of vibration. Under this assumption the character of the nominal dynamics is unaffected by the imposed motion (in the linearized setting) and the effect of the rotation is completely captured by the nodal precession. At rotation rates that are not small with respect to the rate of vibration the dynamics are severely affected and these techniques do not apply. However even at slow rotation rates the effect on the vibration is non-zero. In a future paper we will report on recent work in which we extend the theory of moving systems using Hamiltonian normal form theory to account for the non-adiabatic nature of the imposed motion (see also [2]).

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